

## Note

# A solvable case of image reconstruction in discrete tomography

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## Abstract

A graph-theoretical model is used to show that a special case of image reconstruction problem (with 3 colors) can be solved in polynomial time.

For the general case with 3 colors, the complexity status is open. Here we consider that among the three colors there is one for which the total number of multiple occurrences in a same line (row or column) is bounded by a fixed parameter. There is no assumption on the two remaining colors.

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## 1. Introduction

We shall consider a special case of image reconstruction problem in discrete tomography.

The formulation will be based on graph-theoretic concepts (see [2]) and this will allow us to show that this case can be solved in polynomial time; it generalizes earlier known cases where the problem can be solved. The complexity status of a slight extension of this solvable case is still open; so our result is a step towards the boundary between easy and

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difficult problems in image reconstruction. The reader is referred to [5] for definitions about complexity.

We shall now define the general image reconstruction problem as follows: an image of  $(m \times n)$  pixels of  $p$  different colors has to be reconstructed. For convenience we consider that there is in addition a color  $p + 1$  which is the ground color. We are given the number  $\alpha(i, s)$  of pixels of each color  $s$  in each row  $i$  and also the number  $\beta(j, s)$  of pixels of each color  $s$  in each column  $j$ ; is it possible to reconstruct an image, i.e., can one assign a color  $s$  to each entry  $(i, j)$  of the image in such a way that there are  $\alpha(i, s)$  occurrences of color  $s$  in each row  $i$  and  $\beta(j, s)$  occurrences of color  $s$  in each column  $j$ , for all  $i, j, s$ ?

This simplified version of image reconstruction problems occurring in discrete tomography is denoted by  $R(m, n, p)$ ; it is a combinatorial problem whose complexity status is unknown for  $p = 2$  colors (i.e., when we have  $p + 1 = 3$  colors including the ground color). It is NP-complete for  $p \geq 3$  (see [3,6]). In [4] some special cases solvable in polynomial time have been presented. Notice that it is solvable in polynomial time if  $p + 1 = 2$  (see [7]).

For a solution to exist we must necessarily have

$$\begin{aligned} \sum_{s=1}^{p+1} \alpha(i, s) &= n, \quad i = 1, \dots, m, \\ \sum_{s=1}^{p+1} \beta(j, s) &= m, \quad j = 1, \dots, n, \\ \sum_{i=1}^m \alpha(i, s) &= \sum_{j=1}^n \beta(j, s), \quad s = 1, \dots, p + 1. \end{aligned}$$

These conditions are necessary but not sufficient for the existence of a solution to  $R(m, n, p)$ .

## 2. Graph-theoretical formulation

We associate with the problem a complete bipartite graph  $G = K_{m,n}$  on two sets of nodes  $R, S$  with sizes  $m$  and  $n$ . Each edge  $[i, j]$  of  $K(m, n)$  corresponds to entry  $(i, j)$  in row  $i$  and column  $j$  of the  $(m \times n)$  array.

The image reconstruction problem can be interpreted as follows: The entries of color  $s$  in the array correspond to a subset  $B_s$  of edges (a partial subgraph of  $K(m, n)$ ) such that  $B_s$  has  $\alpha(i, s)$  edges adjacent to node  $i$  of  $R$  and  $\beta(j, s)$  edges adjacent to node  $j$  of  $S$ . We have to find a partition  $B_1, B_2, \dots, B_{p+1}$  of the edge set of  $K(m, n)$  where each  $B_s$  satisfies the above degree requirements.

As mentioned above, for the case  $p + 1 = 3$ , the complexity is unknown. The problem is solvable in polynomial time with  $p + 1 = 4$  colors (see [4]) if three colors, say colors 1, 2 and 3, are *unary*, i.e.  $\alpha(i, s) \leq 1$ ,  $\beta(j, s) \leq 1$  for  $s = 1, 2, 3$ , and all  $i, j$ . In the same paper, it is shown that it is solvable with  $p + 1 = 3$  colors if two colors, say colors 1 and 2, are *semi-unary*, i.e.  $\alpha(i, 1) \leq 1 \forall i$  or  $\beta(j, 1) \leq 1 \forall j$ , and  $\alpha(i, 2) \leq 1 \forall i$  or  $\beta(j, 2) \leq 1 \forall j$ .

Our purpose is to consider an intermediate case between 2 and 3 colors, and show that it can be solved in polynomial time. In particular, the case with  $p + 1 = 3$  colors, one among them being unary, can easily be solved by considering the rows and columns in a specific order.

### 3. A special case of $R(m, n, p + 1 = 3)$

By  $RP3(m, n; q)$  we denote the problem with  $p + 1 = 3$  colors without restrictions on colors 2 and 3, and such that the sum  $s$  of occurrences of color 1, over the lines (rows and columns) where color 1 occurs several times, is bounded by  $q$ ; i.e.

$$s = \sum_{i: \alpha(i, 1) > 1} \alpha(i, 1) + \sum_{j: \beta(j, 1) > 1} \beta(j, 1) \leq q.$$

We shall consider the graph-theoretical formulation of the problem; so we have a complete bipartite graph  $K(m, n)$  and we assume that the nodes  $i$  in  $R$  (corresponding to the rows of the array) are ordered according to the *non-increasing* values of  $\alpha(i, 2)$  and the nodes  $j$  in  $S$  (corresponding to the columns  $j$  of the array) are ordered according to the *non-decreasing* values of  $\beta(j, 2)$ .

If we merge colors 1 and 2, our problem amounts to finding in  $K(m, n)$  a partial graph  $H$  where each node  $i$  in  $R$  has degree  $d_H(i) = \alpha(i, 1) + \alpha(i, 2)$  and each node  $j$  in  $S$  has degree  $d_H(j) = \beta(j, 1) + \beta(j, 2)$ . Such an  $H$  will be called *12-feasible*.

In addition  $H$  must contain in its edge set  $E(H)$  a partial graph  $M$  with degree  $\alpha(i, 1)$  for each node  $i$  in  $R$  and degree  $\beta(j, 1)$  for each node  $j$  in  $S$ . Such an  $M$  will be called *1-feasible*. We shall write  $a < b$  if  $a$  comes before  $b$  in the ordering of the nodes (in  $R$  or in  $S$ ).

We shall say that two edges  $[a, b], [c, d]$  of  $M$  form a *crossing* if  $a < c$ ,  $d < b$  and  $\alpha(a, 1) = \alpha(c, 1) = \beta(d, 1) = \beta(b, 1) = 1$ .

**Lemma 3.1.** *If  $RP3(m, n; q)$  has a solution, then there is also a solution associated to a 1-feasible  $M^*$  which has no crossings.*

**Proof.** Let us assume that there is a crossing  $[a, b], [c, d]$  in a 1-feasible  $M$  contained in a 12-feasible  $H$ .

Notice that if  $[a, d], [c, b]$  are both in  $H - M$  (they are not in  $M$  by definition of a crossing), then we may replace  $[a, b], [c, d]$  in  $M$  by  $[a, d], [c, b]$ ; we get a 1-feasible  $M'$  (which is still contained in  $H$ ) and where the number of crossings has decreased.

Also if  $[a, d], [c, b]$  are both out of  $H$ , we may again replace  $[a, b], [c, d]$  in  $M$  and in  $H$  by  $[a, d], [c, b]$  and we get a new 12-feasible  $H'$  containing a 1-feasible  $M'$  where the number of crossings has decreased.

Let us consider now the case where exactly one of the edges  $[a, d], [c, b]$  is in  $H - M$ ; w.l.o.g. we may assume  $[a, d] \in H - M$  and  $[c, b] \notin H$ . Since the nodes  $j$  in  $S$  are ordered according to the non-decreasing values of  $\beta(j, 2)$  there must be a node  $e$  in  $R$  such that  $[e, b] \in H - M$  and  $[e, d] \notin H - M$ . Notice that  $[e, d]$  cannot be in  $M$  since  $\alpha(d, 1) = 1$ .

So we have  $[e, d] \notin H$ . We replace  $[c, d]$  and  $[e, b]$  in  $H$  by  $[c, b]$  and  $[e, d]$ . Now we replace  $[a, b]$ ,  $[c, d]$  in  $M$  by  $[a, d]$ ,  $[c, b]$  and we get a 1-feasible  $M^*$  contained in a 12-feasible  $H^*$ ; furthermore the number of crossings in  $M^*$  is smaller than in  $M$ .  $\square$

As a consequence of this lemma, the unary case ( $q = 0$ ) has a simple polynomial time solution. First order the vertices as above, assign color 1 without creating a crossing (this assignment is unique), remove the colored edges and color the remaining graph with the usual network flow technique. Notice that the complexity of this algorithm is the same as for the case with  $p + 1 = 2$  colors.

We now define for each instance of a problem  $RP3(m, n; q)$  the class  $\mathcal{C}$  of all 1-feasible partial graphs  $M$  which have no crossings.

$\mathcal{C}$  can be generated as follows: let  $Q$  be the set of lines (rows and columns) which have more than one occurrence of color 1; assume  $m \geq n$ . We first enumerate all partial graphs  $M'$  which satisfy the following:

- (a)  $d_{M'}(i) = \alpha(i, 1)$  for all  $i \in Q \cap R$   
 $\leq \alpha(i, 1)$  for all  $i \in R - Q$ ,
- (b)  $d_{M'}(j) = \beta(j, 1)$  for all  $j \in Q \cap S$   
 $\leq \beta(j, 1)$  for all  $j \in S - Q$ ,
- (c) each edge of  $M'$  has at least one node in  $Q$ .

Since each  $M'$  has at most  $s \leq q$  edges, there are at most  $(m \cdot n)^q \leq m^{2 \cdot q}$  partial graphs to enumerate. For each such  $M'$ , we have to determine the remaining edges to be added in order to obtain a 1-feasible  $M$ . Let  $V'(M')$  be the set of nodes which are not saturated yet (i.e. where some edges of color 1 should be added). We notice that each node in  $V'(M')$  has to receive exactly one edge of color 1. If the basic conditions  $\sum (\alpha(i, 1) \mid i = 1, \dots, m) = \sum (\beta(j, 1) \mid j = 1, \dots, n)$  are satisfied we will have  $|R \cap V'(M')| = |S \cap V'(M')|$ . So there will exist an assignment of color 1 for edges between  $R \cap V'(M')$  and  $S \cap V'(M')$  (because we still have a complete bipartite graph between these two sets of nodes) and there is a unique way of choosing those edges without introducing crossings. So we can obtain a 1-feasible  $M$  from each  $M'$ .

Observe furthermore that any 1-feasible  $M^*$  which has no crossing is in  $\mathcal{C}$ ; this can be seen as follows. We remove from  $M^*$  all edges which are adjacent to some node  $i \in R$  (with  $\alpha(i, 1) > 1$ ) or  $j \in S$  (with  $\beta(j, 1) > 1$ ). We also remove those nodes; this subset of edges removed has been considered as an  $M'$  in the enumeration process. Now the edges remaining in  $M^*$  do not have any crossing; so they are uniquely defined in the complete bipartite subgraph constructed on the remaining nodes. Hence this set  $M^*$  has been constructed in  $\mathcal{C}$ .

We can now state:

**Proposition 3.1.**  *$RP3(m, n; q)$  can be solved in polynomial time.*

**Proof.** We construct each 1-feasible  $M^*$  in  $\mathcal{C}$ ; there are at most  $m^{2 \cdot q}$  such partial graphs. Each one is constructed in  $O(m)$ . Then for each  $M^*$  we do the following: remove  $M^*$  from  $K(m, n)$ ; in the remaining graph  $G^*$  examine if there exists a partial graph  $I$  with  $d_I(i) = \alpha(i, 2)$  for each  $i \in R$  and  $d_I(j) = \beta(j, 2)$  for each  $j \in S$ .

| $i \backslash s$ | 1 | 2 | 3 |
|------------------|---|---|---|
| 1                | 1 | 1 | 0 |
| 2                | 1 | 1 | 0 |
| 3                | 1 | 0 | 1 |

$\alpha(i, s)$

| $j \backslash s$ | 1 | 2 | 3 |
|------------------|---|---|---|
| 1                | 2 | 1 | 0 |
| 2                | 1 | 1 | 1 |

$\beta(j, s)$

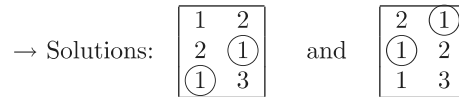


Fig. 1. An example where a generalized crossing cannot be avoided.

If there is such an  $I$ , it gives the assignment of color 2 while  $M^*$  gives the assignment of color 1. So we have found a solution.

If for every  $M^*$  in  $\mathcal{C}$ , no  $I$  can be found, the problem has no solution: according to Lemma 3.1, if there is a solution, there is one when  $M^*$  has no crossing. Since  $\mathcal{C}$  contains all 1-feasible  $M^*$  which have no crossing, we are done.

The construction of  $I$  is a flow problem; so it can be performed in polynomial time ( $O(m^3)$  for instance); see [1].  $\square$

#### 4. Concluding remarks

Removing the assumption on color 1 would give us the general case  $R(m, n, p + 1 = 3)$  whose complexity status is open. In the case where  $\alpha(i, 1) \leq 1$  and  $\beta(j, 1) \leq 1$  for all  $i, j$ , we have some structure in  $M^*$  (no crossings) and an easy construction procedure can be derived.

Notice that in the case studied here we cannot relax the definition of crossings by dropping the equalities  $\alpha(a, 1) = \dots = \beta(b, 1) = 1$ . Such a situation, where a generalized crossing cannot be avoided, is illustrated by the example ( $m = 3, n = 2, p + 1 = 3$ ) given in Fig. 1.

As displayed, the only two ways of coloring the image contain a crossing. Consequently, our method cannot be directly extended to the semi-unary case, nor to the case where  $\alpha(i, 1) \leq 2$  and  $\beta(j, 1) \leq 2$ . However, the technique proposed should be modified and expanded in an appropriate way to tackle those instances and further generalizations.

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